1 Introduction to Finite Difference Schemes

A *finite difference scheme* is an approximation to a continuous derivative which can be implemented as a numerical calculation. For simplicity, let $f(x)$ be a *nice* function, e.g. smooth and continuous, and let $\Delta x \ll x$. A Taylor series expansion of a function is a method of approximating a *nice* function at a point $f(x + \Delta x)$ in terms of its value at a nearby point $f(x)$. The Taylor series of f at $x \pm \Delta x$ is defined as the infinite series

$$f(x \pm \Delta x) = f(x) \pm f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2!} \pm f'''(x)\frac{\Delta x^3}{3!} + \cdots \tag{1}$$

where f', f'', and f''' are shorthand for the first, second, and third derivative of f. For $\Delta x \ll x$, each higher term is smaller than the previous term, therefore it is safe to truncate the series after a few terms. Rearranging (1) produces

$$f(x + \Delta x) - f(x) \approx f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2!} \, . \tag{2}$$

Dividing (2) by Δx produces

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \approx f'(x) + f''(x)\frac{\Delta x}{2!} \, . \tag{3}$$

From (3) a finite difference scheme for the first derivative can be defined

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4}$$

The truncation error of (4) can be obtained from (3) after rearranging terms

$$\left| \frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x) \right| = |f''(x)|\frac{\Delta x}{2!} \, . \tag{5}$$

Since the error for (4) is linear in Δx, the scheme is then refered to as a *first order method*. The corresponding shorthand notation is $\mathcal{O}(\Delta x)$ for term of *order delta x*.

The complement to (4) is

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x} \, . \tag{6}$$

Like the previous scheme, this one is first order accurate.

The final method to be considered is the *center difference* scheme. Taking (4) and (6) and averaging the two schemes together, produces

$$\begin{aligned} f'(x) &\approx \frac{\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{f(x) - f(x-\Delta x)}{\Delta x}}{2} \\ &= \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \end{aligned} \tag{7}$$

The truncation error for the center difference scheme is left as an exercise.